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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematics.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

The Mathematics Situation in Louisiana

The State Department of Education of Louisiana mailed out on May 25, 1938, to superintendents, supervisors, and principals of Louisiana high schools for white students four questions. The first two were as follows:

(1) Should the State Board of Education require of all high school graduates more than one unit in mathematics?
(2) If you favor requiring all graduates to offer two units in mathematics in what subject should this credit be required?

The following is extracted from the tabulated returns of the questionnaire as found in Circular No. 1036:

> Of 283 principals sending in answers to question No. 1, 82 favored one unit only of required mathematics and 201 favored two units. Of 43 superintendents replying, 27 were in favor of two units and 16 favored one unit. Of 33 supervisors replying 12 favored two units and 21 one unit.

In his interpretation of the returns, Supervisor Coxe chose to attach greater weight to the verdict of the 21 supervisors, who favored no change from a one-unit requirement, than to the verdict of the 228 superintendents and principals, who, by much more than a two-to-one majority, favored increasing the requirement to two units.

The fact that the amount of student enrollment under the 98 principals and supervisors who favored one unit of mathematics was approximately equal to that under the 228 principals and superintendents who favored two units of mathematics, could not constitute a ground for interpreting the returns as favoring a one-unit policy. For equality of student enrollments represented could have no relation to the correctness or the incorrectness of the judgments of the 283 principals who were called upon, in effect, to answer the question: Are two units of mathematics better for the student than one unit of mathematics?

Again, because, 42.5% of all those desiring a two-unit mathematics requirement, favored a combination of algebra and arithmetic, 33% a combination of algebra and geometry, and 23.8% favored other combinations, so that there was no majority in favor of any one combination, our Supervisor concluded that a two-unit requirement in mathematics should not be adopted in the high schools of this state.

Comment: Thus, while the 228 ayes for two units of mathematics were given less weight than the 98 nays against two units in mathematics, in his interpretation of returns to question 2, the Supervisor did not see fit, as he had done in question 1, to discount the majority rule. On the contrary, the majority idea was a factor in his judgment when he said: "This lack of agreement illustrates the futility of having any great number agree on the prescriptions for high school graduation."

S. T. Sanders.

The Exposition

The above title was suggested to the writer by a South American student some years ago. It was used in its pure Latin signification, namely, "placing out, or putting forth", or in its more modern or almost slang meaning, "getting across".

The Brazilian government sent a young Roman Catholic priest to Rensselær, to study mathematics and chemistry. It was my privilege to have him in his mathematical studies. He was gifted with keen intelligence, and great earnestness. One day he turned to me, saying, "Professor, the most important quality in the teacher is the 'Exposition'." Many teachers understand the subject, but they cannot "expose" well. And when the teacher exposes the subject he *must remember* that *in front is ignorance.* That remark has remained in my mind ever since. How tremendously full of truth it really is!

To get the topic across, or to explain it, so that it may be clearly understood by all, is the most important duty of the teacher. Nothing which the teacher does is nearly as essential as this ability to "expose" well. After a teacher has taught the same subject for a few years, he finds that all its difficulties have disappeared, because repetition has made them less and less evident. Therefore he is likely to forget that the subject may be anything but clear to the student. Hence the appropriateness of the injunction which terminates the remark of the Brazilian student, "When the teacher exposes the subject, he must remember that *in front is ignorance*". That is just the secret of successful teaching, namely, to remember that each new class contains approximately the same sort of pupils as the last, and that the students are as ignorant as ever, even though to the teacher the subject has become very simple. Failure to remember this important fact leads to much unsatisfactory teaching.

The young teacher will do well to realize that he gains much in the estimate of the student, by great willingness to answer even what might seem like trivial questions.

Also enthusiasm and joy on the part of the teacher are most helpful in arousing interest, and hence in bringing about successful results on the part of the students. There has always been a more or less evident distaste for mathematics shown by the average student, and it has been considered by many as only necessary drudgery in order to gain a passing mark.

This was once well expressed by a fine student after the teacher had given an original proof of a certain topic, and had turned to the class with the query, "There, isn't that a great deal prettier than the proof given in the text?"

The prompt reply of the student was, "No, Professor, but your proof is less ugly than that in the book". Of course the reply was given jokingly, but it shows the influence of the idea that mathematics is dry and dull and ugly, whereas in reality it is the most beautiful of the sciences in many ways.

What greater joy ever comes to man than the feeling of satisfaction after solving a difficult problem over which he has worked perhaps for days or even for weeks.

The object of this contribution is to impress upon the mind of the young teacher the great opportunity which he has in his lifework, if only he will endeavor to "expose" well, and inject into all his work and into his life action the enthusiasm and joy which will surely come to him if he believes and really practices the words of advice here given.

Of course in graduate work it may be considered unimportant how the student reacts to the lecture of the teacher, but undergraduates are "babes in the wood", in mathematical lore, and must be treated as such, by helpful, stimulating eagerness on the part of the instructor.

JAMES MCGIFFERT.

Some Functions Analogous to Trigonometric and Hyperbolic Functions

By V. B. TEMPLE
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§I. *Introduction.*

L. E. Ward* has shown that functions can be formed based on the solutions of the differential equation

$$\frac{d^3u}{dx^3}+u=0,$$

which are analogous to the trigonometric functions sin x and cos x. It is the purpose of this paper to extend the discussion to the most general case and to include, also, a general discussion of functions analogous to the hyperbolic functions sinh x and cosh x.

As a basis for the introduction of this discussion we shall begin with the simplest of these functions, based on the solutions of the differential equations

$$(1) \qquad \frac{d^2u}{dx^2}+u=0 \quad \text{and} \quad (2) \qquad \frac{d^2u}{dx^2}-u=0.$$

Each of these equations has two linearly independent solutions, and from these solutions we can form the trigonometric and hyperbolic functions cos x and cosh x, respectively.

Cos x and sin x. From the solutions of (1) we form the function cos x in the following well-known manner.:

The solutions are

$$u=e^{e_1x} \text{ and } u=e^{e_2x}, \quad \text{where } e_1=i,\ e_2=-i.$$

If we define $u_1(x)$ to be one-half the sum of these solutions, we have cos x. It is

$$(3) \qquad u_1(x)=\tfrac{1}{2}(e^{e_1x}+e^{e_2x}).$$

Its negative derivative we shall define as $u_2(x)$. It is

$$(4) \qquad u_2(x)=-\tfrac{1}{2}(e_1e^{e_1x}+e_2e^{e_2x}).$$

*Ward, *Some Functions Analogous to Trigonometric Functions, American Mathematical Monthly*, Vol. 34 (1927), p. 301.

If we differentiate $u_2(x)$ with respect to x, we have $u_1(x)$ and hence no new functions are obtained by further differentiations. Now, $u_2(x)$ is readily observed to be sin x.

The well-known identity $\sin^2 x+\cos^2 x\equiv 1$ is expressed by the determinant

$$W=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{vmatrix} \equiv 1. \tag{5}$$

We write the addition and double variable theorems as

$$u_1(x+y)=u_1(x)u_1(y)-u_2(x)u_2(y), \tag{6}$$

and on differentiating $u_1(x+y)$ partially with respect to x, we have

$$u_2(x+y)=u_2(x)u_1(y)+u_1(x)u_2(y). \tag{7}$$

Now on replacing y by x, we have

$$u_1(2x)=u_1^2(x)-u_2^2(x), \text{ and} \tag{8}$$

$$u_2(2x)=2u_1(x)u_2(x). \tag{9}$$

Cosh x and sinh x. From the solutions of equation (2) we form the function cosh x as follows:

The solutions of (2) are

$$u=e^{e_1x} \text{ and } u=e^{e_2x}, \quad \text{where } e_1=1,\ e_2=-1.$$

If we define $u_1(x)$ to be one-half the sum of these solutions, we have cosh x. It is

$$u_1(x)=\tfrac{1}{2}(e^{e_1x}+e^{e_2x}). \tag{10}$$

We define $u_2(x)$ as the derivative of $u_1(x)$. It is

$$u_2(x)=\tfrac{1}{2}(e_1e^{e_1x}+e_2e^{e_2x}). \tag{11}$$

If we differentiate $u_2(x)$ with respect to x, we have $u_1(x)$ and hence no new functions are obtained by further differentiations. The reader will readily observe that $u_2(x)$ is sinh x.

The identity $\cosh^2 x-\sinh^2 x\equiv 1$ is expressed by the determinant

$$W=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ u_2 & u_1 \end{vmatrix} \equiv 1. \tag{12}$$

The addition and double variable theorems are

$$u_1(x+y)=u_1(x)u_1(y)+u_2(x)u_2(y), \tag{13}$$

and on differentiating $u_1(x+y)$ partially with respect to x, we have

$$u_2(x+y)=u_2(x)u_1(y)+u_1(x)u_2(y). \tag{14}$$

Now on replacing y by x, we have

$$u_1(2x)=u_1^2(x)+u_2^2(x), \text{ and} \tag{15}$$

$$u_2(2x)=2u_1(x)u_2(x). \tag{16}$$

§II. *The general case of the trigonometric analogies.*

We shall use as the basis for these functions the nth order differential equation

$$\frac{d^n u}{dx^n}+u=0. \tag{17}$$

There are n linearly independent solutions of equation (17) in the form

$$e^{e_1 x},\ e^{e_2 x},\cdots\cdots\cdots\cdots, e^{e_n x},$$

where $e_1, e_2, \cdots\cdots\cdots, e_n$ are the n nth roots of -1.

We shall now define the functions for the trigonometric analogies. Let

$$u_1(x)=\frac{1}{n}(e^{e_1 x}+e^{e_2 x}+\cdots\cdots\cdots+e^{e_n x}), \tag{18}$$

and let

$$u_2(x)=-u_1'(x),\ u_3(x)=-u_2'(x)=u_1''(x),\cdots\cdots\cdots, \tag{19}$$
$$u_n(x)=(-1)^{n-1}u_1^{(n-1)}(x).$$

Since $u_n'(x)=(-1)^n u_1(x)$, we see that further differentiations produce no new functions. These functions are linearly independent solutions of (17).

The Addition and Double Variable Forms. In order to write the addition and double variable functions which are analogous to $\cos(x+y)$ and $\cos(2x)$, we establish the following theorem:

*Theorem 1. $u_1(x+y)=u_1(x)u_1(y)-(-1)^n \sum_{k=2}^{n} u_k(x)u_{n-k+2}(y).$

Proof: From the definitions of the functions we have

$$u_1(x)u_1(y)=\frac{1}{n^2}\sum_{i,j=1}^{n} e^{e_i x}e^{e_j y},$$

*The form of proof here used is due to W. V. Parker, Louisiana State University.

$$(20) \qquad u_k(x) = (-1)^{k-1}\frac{1}{n}\sum_{i=1}^{n} e_i^{k-1}e^{e_i x},$$

Now,

$$u_{n-k+2}(y) = (-1)^{n-k+1}\frac{1}{n}\sum_{j=1}^{n} e_j^{n-k+1}e^{e_j y}.$$

$$(21) \qquad u_1(x)u_1(y) - (-1)^n \sum_{k=2}^{n} u_k(x)u_{n-k+2}(y)$$

$$= \frac{1}{n^2}\sum_{i,j=1}^{n}(1 - \sum_{k=2}^{n} e_i^{k-1}e_j^{n-k+1})e^{e_i x}e^{e_j y}.$$

If ω is a primitive nth root of unity, we may suppose the e_i's to be arranged so that $e_i = e_1\omega^i$, $i > 1$. We have, now, on taking the expression within the brackets,

$$(22) \qquad 1 - \sum_{k=2}^{n} e_i^{k-1}e_j^{n-k+1} = 1 - e_1^n \sum_{k=2}^{n} \omega^{(i-j)(k-1)}.$$

Now let $p = k-1$ and we have

$$(23) \qquad 1 - e_1^n \sum_{k=2}^{n} \omega^{(i-j)(k-1)} = 1 + \sum_{p=1}^{n-1} \omega^{(i-j)p}$$

$$= \sum_{p=1}^{n} \omega^{(i-j)p} = \begin{cases} 0; \ i \neq j \\ n; \ i = j \end{cases}, \qquad (i-j) < n,$$

and therefore we have finally

$$(24) \qquad u_1(x)u_1(y) - (-1)^n \sum_{k=2}^{n} u_k(x)u_{n-k+2}(y)$$

$$= \frac{1}{n}\sum_{i=1}^{n} e^{e_i(x+y)} = u_1(x+y).$$

We can now obtain $u_k(x)$ by differentiating $u_1(x+y)$ $(k-1)$ times partially with respect to x in accordance with definitions (19).

To derive the forms analogous to $\cos(2x)$, we have only to replace y by x in the above functions. Thus, we get in particular,

$$(25) \qquad u_1(2x) = u_1^2(x) - (-1)^n \sum_{k=2}^{n} u_k(x)u_{n-k+2}(x),$$

and any $u_k(2x)$ may be obtained by differentiating $u_1(2x)$ $(k-1)$ times in accordance with the definitions of the functions.

Boundary Conditions. By making use of the well-known* principle that the sum of the kth powers of the roots of the equation $x_n+1=0$, where k is an integer less than n, is zero, it is easy to establish the following boundary conditions:

$$u_1(0)=1,\ u_2^{(n-1)}(0)=1,\ u_3^{(n-2)}(0)=-1,\cdots\cdots\cdot,\quad u_k^{n-k+1}(0)=(-1)^k,\ k=2,\cdots\cdots\cdot,n. \tag{26}$$

And in all other cases we have

$$u_k^{(n-s+1)}(0)=0,\ s\neq k. \tag{27}$$

In order to write the functions analogous to the identity

$$\cos^2x+\sin^2x\equiv1,$$

we shall establish the following theorem:

Theorem 2. The determinant†

$$W=\begin{vmatrix} u_1 & u_2 & \cdots\cdots\cdots u_n \\ u_1' & u_2' & \cdots\cdots\cdots u_n' \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots\cdots\cdots u_n^{(n-1)} \end{vmatrix}\equiv1.$$

Proof: If we differentiate W with respect to x, we shall have

$$\frac{dW}{dx}=\begin{vmatrix} u_1' & u_2' & \cdots u_n' \\ u_1' & u_2' & \cdots u_n' \\ u_1'' & u_2'' & \cdots u_n'' \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots u_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & \cdots u_n \\ u_1'' & u_2'' & \cdots u_n'' \\ u_1'' & u_2'' & \cdots u_n'' \\ u_1''' & u_2''' & \cdots u_n''' \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots u_n^{(n-1)} \end{vmatrix} + \cdots\cdots\cdots + \begin{vmatrix} u_1 & u_2 & \cdots u_n \\ u_1' & u_2' & \cdots u_n' \\ \cdots & \cdots & \cdots \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots u_n^{(n-2)} \\ u_1^{(n)} & u_2^{(n)} & \cdots u_n^{(n)} \end{vmatrix} \tag{28}$$

*Bocher, *Higher Algebra*, p. 243.

†The Wronskian, Morris and Brown, *Differential Equations*, p. 81. For functions satisfying similar conditions see Muir and Metzler, *Theory of Determinants*, p. 442.

It is clear that in each of these determinants there are two rows either equal or proportional, and therefore

$$\frac{dW}{dx}=0, \text{ and} \tag{29}$$

$$W=\text{constant.}$$

Now, making use of the boundary conditions, we have

$$W=\begin{vmatrix} 1 & 0 & 0 & \cdots\cdots\cdots & 0 \\ 0 & 0 & 0 & \cdots\cdots\cdots & (-1)^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1\cdots\cdots & 0 \\ 0 & 0 & -1 & 0\cdots\cdots & 0 \\ 0 & 1 & 0 & 0\cdots\cdots & 0 \end{vmatrix}\equiv 1. \tag{30}$$

Power Series. The power series in x for $u_1(x)$ can be obtained by use of Maclaurin's Series and it is

$$u_1(x)=1+\sum_{i=1}^{\infty}(-1)^i\frac{x^{in}}{\underline{/in}}. \tag{31}$$

If we differentiate $u_1(x)$ $(k-1)$ times in accordance with definitions (19), we have in general

$$u_k(x)=\sum_{i=1}^{\infty}(-1)^{i+k-1}\frac{x^{in-k+1}}{\underline{/in-k+1}},\quad (k=2,\cdots,n). \tag{32}$$

§III. *The General Case of the Hyperbolic Analogies.*

As the basis for these functions we use the nth order differential equation,

$$\frac{d^nu}{dx^n}-u=0. \tag{33}$$

There are n linearly independent solutions of (33) in the form

$$e^{e_1x},\ e^{e_2x},\cdots\cdots\cdots\cdots,\ e^{e_nx},$$

where $e_1, e_2,\cdots\cdots\cdots\cdots, e_n$ are the n nth roots of 1.

We shall define these functions, in particular,

$$u_1(x)=\frac{1}{n}(e^{e_1x}+e^{e_2x}+\cdots\cdots\cdots\cdots+e^{e_nx}), \tag{34}$$

and in general,

$$u_2(x)=u_1'(x),\ u_3(x)=u_2'(x)=u_1''(x),\cdots\cdots\cdots,\ u_n(x)=u_1^{(n-1)}(x). \tag{35}$$

Since $u_n'(x) = u_1(x)$, further differentiations produce no new functions. These functions are linearly independent and each is a solition of equation (33).

The Addition and Double Variable Forms. In order to write these forms we prove the following theorem.

Theorem 3. $u_1(x+y) = u_1(x)u_1(y) + \sum_{k=2}^{n} u_k(x)u_{n-k+2}(y).$

Proof: From the definitions of the functions, we have

$$u_1(x)u_1(y) = \frac{1}{n^2}\sum_{i,j=1}^{n} e^{e_i x}e^{e_j y},$$

$$u_k(x) = \frac{1}{n}\sum_{i=1}^{n} e_i^{k-1}e^{e_i x}, \tag{36}$$

$$u_{n-k+2}(y) = \frac{1}{n}\sum_{j=1}^{n} e_j^{n-k+1}e^{e_j y}.$$

Then we shall have

$$u_1(x)u_1(y) + \sum_{k=2}^{n} u_k(x)u_{n-k+2}(y) \tag{37}$$

$$= \frac{1}{n^2}\sum_{i,j=1}^{n}\left[1 + \sum_{k=2}^{n} e_i^{k-1}e_j^{n-k+1}\right]e^{e_i x}e^{e_j y}.$$

We may now write $e_i = \omega^i$ where ω is a primitive nth root of unity, and on taking the expression within the brackets, we have

$$1 + \sum_{k=2}^{n} e_i^{k-1}e_j^{n-k+1} = 1 + \sum_{k=2}^{n} \omega^{i(k-1)+j(n-k+1)} \tag{38}$$

$$= 1 + \sum_{k=2}^{n} \omega^{(i-j)(k-1)} = 1 + \sum_{p=1}^{n-1} \omega^{(i-j)p},\ p = k-1.$$

Now, the above reduces to

$$\sum_{p=1}^{n} \omega^{(i-j)p} = \begin{cases} n, \text{ if } i=j \\ 0, \text{ if } i\neq j \end{cases}, \quad (i-j) < n, \tag{39}$$

and therefore we have finally

$$(40) \qquad u_1(x)u_1(y)+\sum_{k=2}^{n} u_k(x)u_{n-k+2}(y)$$

$$=\frac{1}{n}\sum_{i=1}^{n} e^{\epsilon_i(x+y)}=u_1(x+y).$$

We can now obtain $u_k(x+y)$ by differentiating $u_1(x+y)$ partially with respect to x (or y) $(k-1)$ times in accordance with definitions (35).

To derive the forms analogous to cosh $(2x)$, we have merely to replace y by x in the above functions. Thus, in particular,

$$(41) \qquad u_1(2x)=u_1^2(x)+\sum_{k=2}^{n} u_k(x)u_{n-k+2}(x),$$

and we can obtain $u_k(2x)$ by differentiating $u_1(2x)$ partially with respect to x $(k-1)$ times in accordance with definitions (35).

Boundary conditions. The boundary conditions for these functions are found in a manner similar to that used for getting the boundary conditions in the trigonometric case. They are, for the hyperbolic functions, as follows:

$$(42) \qquad u_1(0)=1,\ u_2^{(n-1)}(0)=1,\ u_3^{(n-2)}(0)=1, \cdots\cdots\cdots\cdots,$$

$$u_k^{(n-k+1)}(0)=1,\ k=2, \cdots\cdots\cdots\cdots, n,$$

and in all other cases

$$(43) \qquad u_k^{(n-s+1)}(0)=0, \quad s\neq k.$$

To find the functions analogous to the identity $\cosh^2 x-\sinh^2 x\equiv 1$, we establish the following theorem.

Theorem 4. The determinant*

$$W=\begin{vmatrix} u_1 & u_2 & \cdots\cdot u_n \\ u_1' & u_2' & \cdots\cdot u_n' \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots\cdot u_n^{(n-1)} \end{vmatrix} \equiv(-1)^{\frac{(n-2)(n-1)}{2}}.$$

Proof: If we differentiate W with respect to x, we have

*See footnote under theorem 2.

$$(44)\quad \frac{dW}{dx}=\begin{vmatrix} u_1' & u_2' & \cdots u_n' \\ u_1' & u_2' & \cdots u_n' \\ u_1'' & u_2'' & \cdots u_n'' \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots u_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & \cdots u_n \\ u_1'' & u_2'' & \cdots u_n'' \\ u_1'' & u_2'' & \cdots u_n'' \\ u_1''' & u_2''' & \cdots u_n''' \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots u_n^{(n-1)} \end{vmatrix} + \cdots\cdots + \begin{vmatrix} u_1 & u_2 & \cdots u_n \\ u_1' & u_2' & \cdots u_n' \\ \cdots & \cdots & \cdots \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots u_n^{(n-2)} \\ u_1^{(n)} & u_2^{(n)} & \cdots u_n^{(n)} \end{vmatrix}.$$

Since in each of these determinants there are two equal rows,

$$(45)\quad \frac{dW}{dx}=0, \text{ and}$$

$$W=\text{constant}.$$

Then, on placing $x=0$, we have

$$(46)\quad W=\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots\cdots & 0 & 0 \end{vmatrix} \equiv (-1)^{\frac{(n-2)(n-1)}{2}}.$$

Power Series. We can obtain the power series in x by use of Maclaurin's Series, and they are as follows. In particular,

$$47)\quad u_1(x)=1+\sum_{i=1}^{\infty}\frac{x^{in}}{\underline{/\,in}},$$

and in general, we have by differentiating $u_1(x)$ $(k-1)$ times according to definitions (35),

$$(48)\quad u_k(x)=\sum_{i=1}^{\infty}\frac{x^{in-k+1}}{\underline{/\,in-k+1}},\quad (k=2,\cdots,n).$$

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

A First Lesson in the History of Mathematics

By G. A. Miller
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1. *One-half.* The word one-half does not have any direct connection with the word two while the words one-third, one-fourth, etc., are directly connected with the words for the related integers and exhibit a systematic structure. Similarly, the Latin word *semis*, the French word *moitié* and the German word *halb* are not directly connected with the words for the number two in these languages. This suggests an important question in the history of elementary mathematics; viz., did the names for integers arise before those for the corresponding common fractions? One conclusion seems clear; viz., we have no right to assume that "The natural numbers seem to have served the purposes of the world until about the beginning of the historic period".* One of the primary facts of the history of mathematics is that *relatively few of the fundamental questions in which we are likely to become interested can be answered now.*

It should be emphasized that the history of mathematics based on literary evidences is confined to comparatively recent times, extending backward at the present time only to about 4000 B. C., and that very little is now known in regard to mathematical developments before 2500 B. C. In 1934 O. Neugebauer published a volume under the title *Vorlesungen über Geschichte der Antiken Mathematischen Wissenschaften* in which he provided a large number of documentary evidences, especially in regard to the ancient Babylonian and the ancient Egyptian mathematics, and pointed out our present lack of definite knowledge relating to the ancient Chinese and the ancient Hindu mathematics. Unfortunately much has been transmitted which does not rest on a secure basis.

*D. E. Smith, *History of Mathematics*, Vol. 2, p. 208.

Our modern number symbol for one-half ($^1/_2$) is directly connected with that for the number 2 and is in accord with a systematic notation, but in some of the ancient languages the symbol for one-half is quite exceptional. This suggests that the symbols of various fractions arose as individuals before a systematic structure for such numbers had been adopted. There is no evidence which implies that any of the ancient peoples regarded the totality of the positive rational numbers as a connected system. This view may have had a geometric origin and was not commonly held before the number system was associated with the points on a line and general number symbols began to be used both for known and for unknown numbers. Traces of this use appear in the literature of the ancient Greeks but this use did not become general before modern times and it is closely connected with the invention of analytic geometry by R. Descartes (1596-1650) and others.

Even R. Descartes failed to see that negative numbers decrease as their absolute values increase and hence he failed to see that such numbers form a continuous system with the positive numbers. L. Euler (1707-1793) first directed attention to the fact that one passes from positive to negative numbers at two points; viz., by passing through 0 and by passing through ∞. In making such definite historical statements it should be observed that they are made in this form for convenience. In general, historical conclusions should be regarded as working hypotheses only, and one should be continually ready to consider additional evidences. A similar fact in the history of ancient mathematics is that the ancient Egyptians did not usually employ a notation for common fractions whose numerators are different from unity, with the exception of the fraction $^2/_3$. In early times they used our now common multiplication symbol $\times$ to represent $^1/_4$, and this may possibly be the oldest mathematical use of the symbol. It was also used sometimes by the Hindus to represent 4.

A primary fact of the history of mathematics is that the developments in elementary mathematics were affected much more by looking backward than by looking forward in the sense that general rules and homogeneity were an outgrowth of special developments. These special developments were frequently inspired by the immediate needs. Thus the names of various integers and of various fractions were at first used also for implements of measure, or for monetary units. The thought of a connected number system which would be convenient in the more advanced stages of civilization naturally did not dominate the early development of sych a system. Hence these early developments are very complex from the modern point of view. The history of mathematics should not be regarded as an isolated subject but should

be studied from the standpoint of the evolution of human intelligence and it forms an essential part of the study of this evolution.

The slowness with which the elementary operations with rational numbers were developed cannot be understood unless we are able to comprehend the great simplifications due to the use of general rules which evolved slowly from a very complex situation created by dealing with special cases almost exclusively. The main objectives of pre-Grecian mathematics center on the effective use of positive rational numbers with respect to the fundamental operations of arithmetic. It has recently been discovered that the ancient Babylonians were also interested in algebraic operations and found roots of various equations, especially of equations of the second degree and of equations which are in the form of such equations. No instance has yet been found in which they found a negative root of such an equation, although they occasionally used negative numbers.

2. *Sexagesimal system.* The face of a clock exhibits a large number of implications relating to the history of elementary mathematics. The division of the hour into 60 minutes and of the minute into 60 seconds raises questions whose study naturally carries us backward through thousands of years to the sexagesimal sy tem of the ancient Sumerians and the ancient Babylonians. Their use of the same symbol for all the numbers whose ratio to each other is a power of 60 naturally arouses our curiosity. The fact that they used a partially developed positional system but failed to employ a symbol corresponding to our decimal period may lead us to consider the advantages of our decimal period and the undesirable changes in our calculations if we had to dispense with its use. It is somewhat difficult to determine the implitions involved in the fact that in the most advanced pre-Grecian mathematical civilizations there was used a numerical notation with such obvious defects.

Some of these defects are partially explained by the fact that numerical values depend upon the units of measure which are employed. For instance, 10 in dollars is equivalent to 1000 in cents. It should also be noted that the sexagesimal system of the ancient Babylonians enabled them practically to dispense with the use of common fractions. For instance, $^1/_2$, $^1/_3$, $^1/_4$, $^1/_5$ and $^1/_6$ were usually represented by them as 30, 20, 15, 12 and 10 respectively. The fraction a/b in which b is divisible by no prime number, when reduced to its lowest terms, except 2, 3 or 5, could be represented by them as an integer and they had numerous tables which simplified the use of such fractions. It is not implied that their system would be satisfactory to us, but it had many advantages in an early stage of mathematical attainments and is unique in the history of our subject.

It has sometimes been inferred that the ancient Babylonians employed 59 different number symbols for the numbers less than the base of their system, just as we use 9 symbols for the digits. A tatement to this effect appears on page 21 of such a widely used work of reference as the *Encyklopädie der mathematischen Wissenschaften*, volume I (1898). There is however no historical evidence for it. In fact, the ancient Babylonians represented the natural numbers which are less than 60 by the use of he base 10 and usually employed only two distinct number symbols in their representation of numbers. It should be emphasized that our number system to the base 10, with its nine different symbols for the digits less than its base, is the only completely developed number system in the history of mathematics which has been widely used. The other systems thus used were only partially developed.

There naturally was a long period during which only a few integers and only a few fractions were used as individuals before a general method of structure and a general system was adopted. Most of the evidences of special individual names and symbols have doubtless been lost but enough remain to support this reasonable hypothesis. Number systems seem to be due mainly to practical use and not to theoretical considerations of those who looked far ahead of the needs of their times. This practical element plays a very fundamental rôle in the development of mathematics, not only in ancient times but also much later. For instance, negative numbers were generally adopted as a result of their persistent use by those who had not fully established the fact that they are theoretically legitimate numbers of our number system. As late as the eighteenth century there were prominent mathematicians who opposed the use of negative numbers.

It should be emphasized that mathematical history differs widely from mathematical tradition. Among the traditions which have recently been frequently stated as facts is the one that the ancient Egyptians constructed right angles by means of a rope with knots separated by distances in the proportion of 3,4,5. This legend seems to have been started by the noted German mathematical historian Moritz Cantor (1829-1920) and it was due to a misinterpretation. It was widely adopted by other writers as a result of the great influence of Moritz Cantor along the line of the history of mathematics. In recent years it has been found that he made a large number of other mistakes and hence his reputation has declined. In view of the rapid growth in historical knowledge during recent years many of the statements in some of the older works are no longer in accord with the present views.

An interesting element of the sexagesimal system is that it is still widely used as a secondary system. Our tables in trigonometry are

still to a considerable extent based on it. It seems to have been preceded by a decimal system even in Sumeria and in ancient Babylon, and to have been largely adopted in Greece about 200 B. C., but it failed to become as dominant among the Greeks as it had been among the ancient Babylonians. In the countries of western Europe and in America it has never played a very prominent rôle, but it has persisted, and the efforts to replace it in the measurement of angle and in the tables used in trigonometry have not as yet met with complete success. Our decimal fractions have largely supplanted its functions in regard to the avoidance of the use of common fractions even if the base 10 is less useful than the base 60 in this regard. At any rate, the fact that it has persisted as a secondary system with more or less prominence is of great historical interest and naturally arouses our interest in its merits and in the fact that it became the dominant system in the most advanced pre-Grecian mathematical civilization.

3. *Abstractions.* Mathematics is largely an abstract science and hence one of the most fundamental questions in the history of mathematics is when mathematics became an abstract science. As the oldest mathematical literature is largely abstract, and relatively about as largely abstract as the modern mathematical literature, it is impossible to answer this question now. The numbers 1,2,3, etc., are universally regarded as abstract. This is also true as regards the common fractions and the sexagesimal fractions. Hence the ancient Babylonian and the ancient Egyptian tables which have been preserved deal largely with abstract mathematics although there are also ancient tables which deal with concrete numbers. In much of the mathematical literature both abstract and concrete numbers appear and hence there is frequently no clear line of demarcation between the abstract and the concrete mathematical literature.

In view of the fact that Webster's *New International Dictionary* is widely used as a work of reference it may be desirable to refer in this connection to the following statement, which appears therein (1938) under the term "algebra". "The essential difference between arithmetic and algebra is that the former deals with concrete quantities while the latter deals with symbols whose values may be any out of a given number field". As the term "arithmetic" is now commonly understood it deals largely with abstract numbers both in its elementary part and also in its more advanced parts and there is no essential difference between it and algebra. At any rate, it is by no means confined to the treatment of concrete quantities, and it would be difficult to prove that it deals relatively more with concrete quantities than algebra does,

since there is now no generally accepted line of demarcation between these subjects.

The early use of abstract numbers illustrates a primary fact of the history of mathematics which may be formulated as follows: *Many of the fundamental advances in mathematics were not premeditated by men of unusual foresight but resulted from the practices of many who improved a little here and there on the work of their predecessors without knowing that they were contributing thereby towards the advancement of an important subject.* Even now children frequently speak abstractly but think concretely as, for instance, when they first learn the common multiplication table. Abstract numbers are used for their convenience long before their significance is fully understood. From the facts that four times three men are twelve men and four times three horses are twelve horses, etc., the child finds it natural to say that four times three are twelve. One of the motives for the development of abstract mathematics is that it provides economy of thoughts and words.

Abstract mathematics is sometimes called pure mathematics. The subjects of pure and applied mathematics have supplemented each other from the time of the earliest mathematical literature up to the present, and it would be difficult to determine the relative extent of these two subjects or the relative amounts of their iteratures. It has been frequently assumed that the ancient Babylonians were led to the study of mathematics by their interest in astronomy but the recent deciphering of much of their mathematical literature by O. Neugebauer and others has led to the conclusion that their mathematical texts extend a full thousand years further back than their systematic astronomy. As far as available literary evidences furnish a basis of judgment pure mathematics is as old as applied mathematics.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Note on the Projection of Three Collinear Points Into Three Other Collinear Points

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It is customary in texts on plane projective geometry to prove that any three points on a straight line can be projected into any other three points on a straight line by no more than two projections if the lines are distinct, or by no more than three projections if the lines coincide. The proofs consist in showing how the respective constructions can actually be carried out, and are therefore quite satisfying, especially to beginning students. However they have this drawback, that the constructions they exhibit, while always possible, are by no means entirely general. Thus in the usual construction for a projectivity sending three points $A,B,C,$ on a line $L,$ into three points $A',B',C',$ on a second line $L',$ the centers O and O' of the two projections are chosen arbitrarily *on one of the three lines* $AA',BB',CC',$ say $AA',$ which join pairs of corresponding points. When this is done the axis, $\bar{L},$ is uniquely determined as the join of the points $\bar{B}$ and $\bar{C}$ in which the lines OB and OC meet $O'B'$ and $O'C'$ respectively. $\bar{A}$ is of course the intersection of AA' and $\bar{L}$. This is evidently a special case of the following theorem:

Theorem I. Given the points $A,B,C,$ on $L,$ and the corresponding points $A',B',C',$ on L'. These two sets of points can always be projected one into the other by two (real) projections whose centers may be chosen as follows: The first may be taken arbitrarily outside* the conic K which is tangent to the five lines $AA',BB',CC',$ L and L'. The second may be chosen arbitrarily on either of the tangents T_1 or T_2

*By the outside of a conic is meant the region from which real tangents can be drawn to the curve.

which can be drawn to the conic K from the first point. When the centers have been chosen the axis of the projectivity is uniquely determined.

This result is an immediate consequence of two well-known theorems: I. A projective correspondence is uniquely determined if three pairs of corresponding elements are given. II. The lines joining corresponding points in a projective correspondence between two lines is a conic tangent to the two lines. Suppose in fact that O, the center of the first projection, is chosen arbitrarily outside the conic K. Let the tangents T_1 and T_2 which can be drawn to K from O meet L and L' in points P_1,P_1' and P_2,P_2' respectively. Then from the theorems cited above, P_1,P_1' and P_2,P_2' are pairs of corresponding points in the projectivity determined by the pairs AA',BB',CC', and moreover a construction sending P_1 into P_1', and P_2 into P_2', and one of the given points into its correspondent, say A into A', will necessarily send the other two, B and C, into their proper images B' and C'. But when the three pairs P_1,P_1'; P_2,P_2'; A,A'; are considered we have exactly the situation treated in the conventional proof. The center O of the first projection is on both T_1 and T_2 each of which is a line joining two corresponding points, and hence O', the center of the second projection can be chosen arbitrarily on either of these lines, the tangents to K from O. In passing we observe that the axes $\bar{L}$ which correspond to O chosen arbitrarily on T_i are the lines of the pencil on the intersection of L' and T_j.

It is interesting to note the relation of this problem to Grassmann's construction of the general cubic curve. If O is considered fixed, the condition which O' must satisfy is that the lines joining it to three fixed points, A',B',C', must intersect three fixed lines OA,OB,OC, in three collinear points. The analytic formulation of this condition is an easy matter and leads to a cubic curve as the locus of O'. This is of course to be expected since the condition on O' is precisely the one employed by Grassmann in his synthesis of the general cubic*. In the present instance, since the three fixed lines are concurrent (in O) and not independent, the cubic is composite, the line bearing A',B',C', being always a component. If O' is taken on this line the solution is trivial, so that the proper locus of O' is a conic. From the preceding paragraph we see that this conic is also composite, consisting of the tangents T_1 and T_2 to K from O, if O is outside K; consisting of the tangent to K at O counted twice, if O is on K; and consisting of two imaginary lines if O is inside K.

*H. S. White, *Plane Cubic Curves*, p. 108.

From the preceding work it is clear that three points on one line can always be projected into three other points on the same line by no more than three projections, simply by projecting the points of one triad from the given line onto any other convenient line, and then applying Theorem I. This is not necessarily the most efficient construction, however, as the following theorem indicates.

Theorem II. Given the points $A,B,C,$ and A',B',C', on L. These two sets can be projected one into the other by two real projections when and only when the fixed points of the correspondence set up on L by the pairs of points AA',BB',CC', are real. In this case the center of the first projection can be selected arbitrarily in the plane, and the center of the second projection can be taken arbitrarily on on either of the lines L_1 or L_2 which join the first center to the fixed points P_1 and P_2 on L. When the centers have been chosen, the axis of the projectivity is uniquely determined.

To establish this theorem we merely observe that the point in which the line joining any possible O and O' meets L must be fixed in the correspondence established on L by the pairs of points AA',BB',CC'. Moreover if O and O' are taken collinear with one of the fixed points P_i, any two of the three given pairs of corresponding points, say AA' and BB', can be used to determine the axis of a projectivity sending A into A',B into B', and sending P_i into itself. In fact the axis $\bar{L}$ will be just the join of $\bar{A}$ and $\bar{B}$, the intersections of OA and OB with the lines $O'A'$ and $O'B'$ respectively. But by the fundamental theorem of projective geometry such a projectivity must necessarily send any point into its proper image, whence in particular C is sent into C'.

If the fixed points on L are real and distinct, the axes of the projectivities corresponding to choices of O and O' on a line through P_i are lines of the pencil on P_j. If the fixed points coincide the axes must pass through this point. If the fixed points are conjugate imaginaries the construction is impossible.

Finally, we observe that the anxiliary constructions implied by Theorems I and II, namely the constructions of the lines which pass through an arbitrary point and touch a conic defined by five of its tangents, and the construction of the fixed points of a correspondence defined on a line by three pairs of corresponding points, are possible second degree constructions.*

*Cremona, *Projective Geometry*, pp. 170, 176.

Mathematical World News

Edited by
L. J. Adams

The annual meeting of the Mathematical Association of England was held on January 2, 3 at King's College in London.

On March 11, 1939 the Liverpool Mathematical Society will meet. Professor E. T. Whittaker will address the Society on the subject *Waves and Particles.*

There is a very interesting article entitled *Impressions of School Mathematics in the United States* by F. J. Wood in the December, 1938 issue of the Mathematical Gazette (England). It is exactly what the title indicates—a description of the impressions of our school mathematics by a visiting Englishman.

The Carus Mathematical Monographs, published by the Mathematical Association of America, are expository treatments of certain fields of mathematics. Those published so far are:

1. *Calculus of Variations.* G. A. Bliss.
2. *Analytic Functions of a Complex Variable.* D. R. Curtiss.
3. *Mathematical Statistics.* H. L. Rietz.
4. *Projective Geometry.* J. W. Young.
5. *History of Mathematics in America Before 1900.* David Eugene Smith and Jekuthiel Ginsburg.

There are four new booklets which will be of interest to those persons who enjoy mathematical recreation:

1. *Boss Puzzle und verwandte Spille.* G. Kowalewski.
2. *Magische Quadrate und Magische Parkette.* G. Kowalewski.
3. *Der Keplersche Korpen und andere Bauspiele.*
4. *Panmagische Quadrate und Sternirelecke.* F. Fitting.

The publishers are K. F. Kœhlers Antiquarium, Leipzig, Taubchenweg 21.

Most mathematics instructors will want to examine *Mathematical Snapshots* by H. Steinhaus (Warsaw). H. Steinhaus is co-editor of *Monografie Matematyczne* and *Fundamenta Mathematicæ*. It includes 135 pages of text, 180 photographs and a pocket containing cards, dodecahedron and red-and-green spectacles. It can be obtained through G. E. Stechert & Co., 31 East 10th Street, New York.

The American Mathematical Society was scheduled to meet at Columbia University on February 25. Professor R. P. Agnew of Cornell University was invited to deliver an address on *Properties of generalized definitions of limit*. Papers were to be presented in three section meetings, devoted to algebra and number theory, analysis and geometry and topology.

On April 7-8 the American Mathematical Society will meet at Durham, North Carolina. The meeting occurs in connection with Duke University's centennial celebration. Invited addresses will include: *Configurations defined by theta functions* by Professor A. B. Coble, *The ergodic theorem* by Professor Horbert Wiener, and *Invarianst* by Professor Hermann Weyl.

Recently there has appeared on the market a low-priced slide rule, manufactured by several different instrument makers in the United States. These new rules do much toward popularizing the use of the slide rule, especially in the secondary schools. They are priced as low as twenty-five cents.

Professor Pauline Sperry, University of California, has been on leave of absence for the past semester.

The Rocky Mountain Section of the M. A. A. will meet at Laramie, Wyoming, on April 28-29.

This department has received a *Catalogue des Oeuvres Scientifiques* from the firm of Nicola Zanichelli in Bologna, Italy. It contains a list of important recent publications of books, monographs and articles by Italian mathematicians. The bibliography is annotated in rather complete, scholarly fashion.

The sixteenth annual meeting of the Indiana Section of the Mathematical Association of America will be held on Friday evening and Saturday, April 28 and 29, 1939, at Ball State Teachers College,

Muncie, Indiana. The annual banquet will be held on Friday evening followed by an address by a prominent speaker. The officers for 1939 are: Professor C. K. Robbins, Purdue, President; Professor L. S. Shively, Ball State, Vice-President; and Professor P. D. Edwards, Ball State, Secretary.

Highlighted by a seminar on the calculus of variations, a broad academic program for the summer quarter has been arranged by the department of mathematics of the University of Chicago, according to plans announced today.

The stress on analysis this summer is in line with a practice of the department during recent years to place emphasis on a particular field each summer quarter. Last summer the focus was on algebra.

While no freshman courses are offered, another feature of the forthcoming summer quarter opening June 21 will be the opportunity for double minors in calculus in courses on Differential Calculus by Dr. Max Coral, of Wayne University during the first term, and Integral Calculus by Dr. Malcolm F. Smiley, of Lehigh University, during the second term.

The seminar on the calculus of variations will be conducted by Gilbert A. Bliss, Martin A. Ryerson distinguished service professor of mathematics and chairman of the department, and a series of lectures preparatory to the special conference on the Calculus of Variations, June 27-30, will be given in the seminar in the preceding week.

Speakers at the conference will include Tibor Rado, Ohio State University, on "Length and area," and "Geometrical approach to the Plateau problem"; Jesse Douglas, formerly of Massachusetts Institute of Technology, on "The problem of Plateau-Riemann" and "Minimal surfaces of higher topological structure".

Edward J. McShane, University of Virginia, on "Existence theorems for multiple integrals"; Dr. Coral on "The equations of Haar and differentiability of their solutions"; Professor Bliss on "The field theory for multiple integrals"; William T. Reid, assistant professor of mathematics at the University, on "Sufficiency proofs by expansion methods"; Marston Morse, Institute for Advanced Study, on "Functional topology and analysis in the large" and "Variational theory in the large"; Magnus R. Hestenes, assistant professor of mathematics at the University, on "The problem of Bolza"; Karl Menger, University of Notre Dame, on "Logical analysis of the semi-continuity properties of line integrals"; and Lawrence M. Graves, associate professor of mathematics at the University, on "The Jacobi condition for multiple integrals".

Professor McShane also will be a member of the University faculty throughout the summer quarter and will present a course on Modern Theories of Integration.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

SOLUTIONS

No. 57. Proposed by *R. B. Thompson*, Beaver Crossing, Nebraska.

Determine a point such that the sum of the nth powers of the distances to three fixed points is a minimum.

Solution by *Johannes Mahrenholz*, Gottbus, Germany.

Let $M(x,y)$ be the required point and let $A,B,C,$ with coordinates (x_i,y_i) $i=1,2,3,$ be the fixed points. Then MA, MB, MC are given by the usual distance relations. It is required that

$$(MA)^n+(MB)^n+(MC)^n$$

be a minimum. Let $f(x,y)$ represent this expression; that is,

1. $$f(x,y)=[(x-x_i)^2+(y-y_i)^2]^{n/2}.$$

The conditions require that

2. $$\delta f/\delta y=\delta f/\delta x=0.$$

If the inclinations of MA, MB, MC are denoted by $\alpha_1,\alpha_2,\alpha_3$, respectively, then the derivatives may be written as:

3. $$\cos\alpha_i\,[(x-x_i)^2+(y-y_i)^2]^{(n-2)/2}=0$$

and $$\sin\alpha_i\,[(x-x_i)^2+(y-y_i)^2]^{(n-2)/2}=0,$$

which may be solved in the following fashion:

$$(MA)^{n-1} : (MB)^{n-1} : (MC)^{n-1}$$
$$=\begin{vmatrix}\cos\alpha_2 & \cos\alpha_3\\ \sin\alpha_2 & \sin\alpha_3\end{vmatrix} : \begin{vmatrix}\cos\alpha_3 & \cos\alpha_1\\ \sin\alpha_3 & \sin\alpha_1\end{vmatrix} : \begin{vmatrix}\cos\alpha_1 & \cos\alpha_2\\ \sin\alpha_1 & \sin\alpha_2\end{vmatrix}$$

or 4. $(MA)^{n-1} : (MB)^{n-1} : (MC)^{n-1}$

$$=\sin(\alpha_3-\alpha_2) : \sin(\alpha_1-\alpha_3) : \sin(\alpha_2-\alpha_1).$$

Recalling the definition of α_i, we have

$$\alpha_3-\alpha_2=\angle BMC(\text{or its supplement})$$

$$\text{so that } \sin(\alpha_3-\alpha_2)=\sin\angle BMC.$$

Likewise, $\sin(\alpha_1-\alpha_3)=\sin\angle CMA$, $\sin(\alpha_2-\alpha_1)=\sin\angle AMB$.

Now the area of ΔBMC is $(MB)(MC)(\sin\angle BMC)/2$. Similarly for the others. Thus (4) becomes:

$$(MA)^{n-1} : (MB)^{n-1} : (MC)^{n-1}$$

$$=(2\Delta BMC)/(MB)(MC):(2\Delta CMA)/(MC)(MA):(2\Delta AMB)/(MA)(MB)$$

$$=(MA)\Delta BMC:(MB)\Delta CMA:(MC)\Delta AMB,$$

or $(MA)^{n-2} : (MB)^{n-2} : (MC)^{n-2}=\Delta BMC : \Delta CMA : \Delta AMB.$

Obviously, for $n=1$, the three angles at M are each 120°; while for $n=2$, $\Delta BMC=\Delta CMA=\Delta AMB$, and M is the centroid of ABC.

Solving for x and y, we find:

$$x=\frac{x_1(MA)^{n-2}+x_2(MB)^{n-2}+x_3(MC)^{n-2}}{(MA)^{n-2}+(MB)^{n-2}+(MC)^{n-2}}$$

$$=\frac{x_1(\Delta BMC)+x_2(\Delta CMA)+x_3(\Delta AMB)}{ABC}$$

$$y=\frac{y_1(\Delta BMC)+y_2(\Delta CMA)+y_3(\Delta AMB)}{ABC}.$$

A discussion of the geometrical features is found in R. Sturm, *Maxima und Minima in der elementaren Geometrie*, Leipsig, (Teubner), 1910, pp. 55-61.

No. 249. Proposed by *W. V. Parker*, Louisiana State University, University, La. 3 ?

If the roots of $f(x)\equiv x^2+px+q=0$ are $\alpha+i\beta$, $\alpha-i\beta$, -2α, where α and β are real and different from O, the roots of $f'(x)=0$ are the points in the complex plane midway between the center and foci of the ellipse with axes on the axes of coordinates and passing through $(\alpha,\beta),(\alpha,-\beta)$ and $(-2\alpha, 0)$.

If the roots of $f(x)=0$ are $\alpha+\beta$, $\alpha-\beta$, -2α, where α and β are real and different from zero, the roots of $f'(x)=0$ are points in the complex plane midway between the center and foci of the hyperbola with axes on the axes of coordinates and passing through $(\alpha, i\beta)$, $(\alpha, -i\beta)$ and $(-2\alpha, 0)$.

Solution by *Fred Marer*, Los Angeles City College.*

By symmetric functions of the roots of $f(x)=0$ we have $p=\beta^2-3\alpha^2$. The ellipse of the form $x^2/A^2+y^2/B^2=1$, passing through the points $(\alpha, \pm\beta)$ and $(-2\alpha, 0)$, requires $A^2=4\alpha^2$, $B^2=4\beta^2/3$. Thus the foci are $(0, \pm 2\sqrt{p/3})$, $(0,0)$, $(\pm 2\sqrt{-p/3}, 0)$ according as $p>0$, $p=0$ or $p<0$, respectively. In any case, the foci are the points associated with the complex numbers $\pm 2\sqrt{-p/3}$. Since the roots of $f'(x)\equiv 3x^2+p=0$ are represented by $\pm\sqrt{-p/3}$, the first proposition is established.

If the above, $i\beta$ be put for β, the ellipse becomes the hyperbola $x^2/4\alpha^2-3y^2/4\beta^2=1$, through the points $(\alpha, \pm i\beta)$ and $(-2\alpha, 0)$. With $p=-\beta^2-3\alpha^2$, the foci are always $(\pm 2\sqrt{-p/3}, 0)$. Hence the conclusion follows as above.

No. 250. Proposed by *Fred Fender*, New Brunswick, N. J.

Of all Pythagorean triangles, which is the one most nearly isosceles with sides less than 10,000?

Solution by *C. W. Trigg*, Los Angeles City College.

It is well known that the two-parameter representation of the legs and hypothenuse of all Pythagorean triangles is

$$a=m^2-n^2,\ b=2mn,\ c=m^2+n^2.$$

I. (a and b approximately equal). Put $m^2-n^2=2mn\pm k$, where k is a positive integer. Since $m>n$, we have $m=n+\sqrt{2n^2\pm k}$ which which equals approximately $n(1+\sqrt{2})$. Thus the maximum value of n is about 38, since $10{,}000>m^2+n^2$ equals approximately $n^2(4+2\sqrt{2})$. We therefore seek the greatest value of $n\leqq 38$ such that $2n^2\pm k$ shall be a perfect square for $k=1, 2, \cdots$. For $k=1$, by the familiar solution of the Pell equation $2n^2\pm 1=(m-n)^2$, we find $n=29$, $m=70$, $a=4059$, $b=4060$, $c=5741$. No further search is necessary since, if $k>1$, the ratio b/a (or a/b if $b<a$) must be greater than $4060/4059$.

*As originally submitted, the solution considered the generalization of the proposition to an equation $f(x)=0$ of degree n, the roots of $f^{(n-j)}(x)=0$, and an "ellipse" $x^2/A^2+y^2/B^2=1$ where x, y, A, and B are any complex numbers.—ED.

II. (a and c approximately equal). The relation

$$m^2+n^2=m^2-n^2+k \text{ requires } n^2=k/2,$$

so that the smallest possible value of k is 2, and thus $n=1$ and $m<100$. Now if $m=99$, we have $a=9800$, $b=198$, $c=9802$.

III. (b and c approximately equal). If in the relation

$$m^2+n^2=2mn+k \text{ we set } k=1,$$

we have $m-n=1$ and $10{,}000>m^2+n^2=2n^2+2n+1$ or $n\leqq 70$. Now if $n=70$, we have $m=71$, $a=141$, $b=9940$, $c=9941$.

Since $1<9941/9940<9802/9800<4060/4059$, there is reason to call the triangle in case III the one most nearly isosceles with sides less than 10,000.

Also solved (case I) by *H. T. R. Aude*, *W. B. Clarke*, *J. Mahrenholz*, *G. W. Wishard* and the *Proposer*.

Editor's Note: The general solution of the Pythagorean triangle whose legs differ by unity was known to Girard in 1625. Fermat stated that if $(x, x+1, z)$ is one such triangle then

$$(2z+3x+1,\ 2z+3x+2,\ 3z+4x+2)$$

is the next larger. Hopkins and Jenkins reduced the problem to the Pell equation

$$A^2-2B^2=\mp 1$$

and gave the recursion formulas for its solution, viz:

$$A_{n+1}=A_n+2B_n,\ B_{n+1}=A_n+B_n,\ A_0=1,\ B_0=0.$$

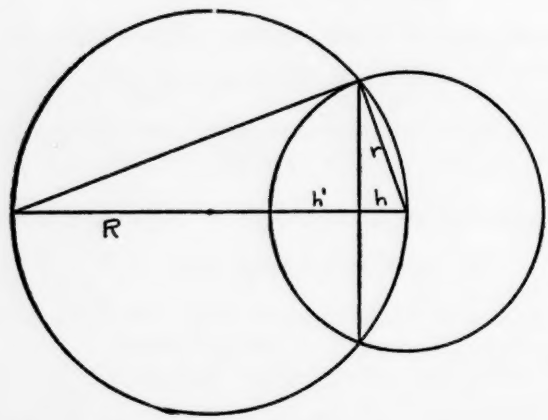
Evans solved the same using the convergents of the continued fraction $\sqrt{2}$.

No. 251. Proposed by *Yudell Luke*, University of Illinois.

A sphere of radius r has its center on the surface of another sphere of radius R with $2R\geq r$. If the area common to both spheres equals one-half the area of the sphere of radius r, find R in terms of r. If R is constant, when is the common area a maximum?

Solution by *C. W. Trigg*, Los Angeles City College.

Strictly speaking, there is no "area common to both spheres". If what is meant by this phrase is "the area of the lens made by the two spheres", denote this area by A. Let h be the height of the zone of



sphere R and h' be the height of the zone of the sphere r which is on the lens. Now if an altitude be dropped to the hypotenuse of the right triangle, either leg is the mean proportional between the hypotenuse and the adjacent segment, so $h=r^2/2R$ and $h'=r-h=r-r^2/2R$.

The area of a zone is $2\pi Rh$, so

$$A=2\pi R(r^2/2R)+2\pi r(r-r^2/2R)=\pi r^2(3-r/R).$$

If $A=\frac{1}{2}(4\pi r^2)=2\pi r^2$ it is already expressed in terms of r. This is equivalent to $r=R$.

If R is constant,

$$\frac{dA}{dr}=\pi\left(6r-\frac{3r^2}{R}\right).$$

When this is equated to zero and solved, $r=0$, for which value A is a minimum; or $r=2R$, which gives the maximum value of $A=4\pi R^2$.

Also solved by the *Proposer*.

No. 252. Proposed by *Walter B. Clarke*, San Jose, California.

P is any point not on the line XY. $A, B, C, D, \ldots$ are points on XY which for convenience are all taken on the same side of the foot of the perpendicular from P to XY. Using an arbitrary angle θ for base angle, similar isosceles triangles are formed on PA, PB, PC, etc., (they may be on either side but all on same side), giving apexes A', B', $C', \ldots$. Show (1) A', B', $C', \ldots$ are collinear; (2) this line makes an angle θ with XY.

Solution by *Henry Dantzig*, University of Maryland.

Let the coordinates of P be $(0,2p)$ and those of A be $(2t,0)$. Let $PAA'=\Theta$ with $\tan\Theta=k$, and the variable angle $OPA=\alpha$. Then, if $PA=2z$, the altitude of the isosceles triangle is kz. If D is the midpoint of the base then the coordinates x,y of A' are the horizontal and vertical projections of the broken line PDA'. That is,

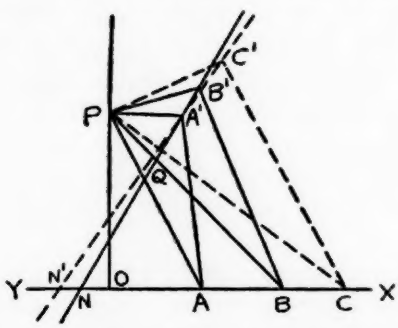
$$\begin{cases} x=t+kz\cdot\cos\alpha=t+kp \\ \\ y=p+kz\cdot\cos\alpha=p+kt. \end{cases}$$

These are the parametric equations of the path of A' as t varies. By an obvious elimination,

$$y=kx+p(1-k^2),$$

a line making an angle Θ with the horizontal.

Solution by *C. W. Trigg*, Los Angeles City College.



It is not necessary that the similar triangles be isosceles, but merely that the angles with vertex P be equal to Θ. Consider two such triangles $PAA'\sim PBB'$. Draw $B'A'$ extended and meeting PB at Q and XY at N. Then $\angle A'PA=\Theta=\angle B'PB$, $\angle A'PB=\angle A'PB$, so $\angle B'PA'=\angle BPA$. Now $PA':PB'::PA:PB$ so $\Delta PA'B'\sim\Delta PAB$, hence $\angle PB'A=\angle PBA$. These angles are also in triangles PQB' and NQB, respectively, which also contain the equal vertical angles PQB' and NQB, hence $\angle A'NX=\angle B'PQ=\Theta$.

In like manner $C'A'N'$, determined by triangle $PA'A$ and any other similar triangle PCC', may be shown to meet XY at an angle $A'N'X=\Theta$. Hence N and N' coincide as do NA' and $N'A'$ and the vertices of the similar triangles are collinear.

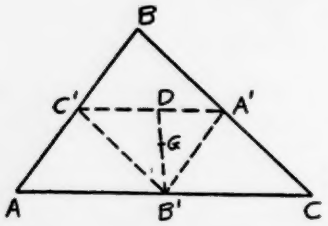
A similar proof is applicable if the triangles are on the other side of the line segments.

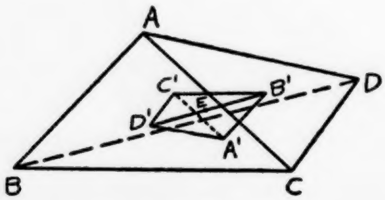
Also solved by *Albert Farnell* and the *Proposer*.

No. 253. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Point out an analogous feature between the center of mass of the perimeter of a triangle and the center of mass of the surface of a tetrahedron.

Solution by *C. W. Trigg*, Los Angeles City College.





(1) The centroids of the sides, AB, BC, CA, of a triangle ABC are at their midpoints, C', A', B', respectively. $A'B'=\frac{1}{2}AB$, $B'C'=\frac{1}{2}BC$, and $C'A'=\frac{1}{2}CA$. The centroid, D, of AB and BC falls on $C'A'$ so that $C'D:DA'::BC:AB::B'C':A'B'$. Hence $B'D$ is the bisector of angle $C'B'A'$. Furthermore, the centroid, G, of the three sides falls on $B'D$. Similarly, G falls on each of the internal angle-bisectors of triangle $A'B'C'$, so G is the incenter of this medial triangle. That is, the center of mass of the perimeter of a triangle is the center of its Spieker circle.*

(2) In the tetrahedron $ABCD$, let the centroid of face ABC be D', the centroid of ACD be B', etc. Now $\Delta A'B'C'=(1/9)\Delta ABC$, and similarly for the other corresponding faces of the tetrahedron and its medial tetrahedron. The centroid, E, of ΔABC and ΔADC lies on $D'B'$ so that $D'E:EB'::\Delta ADC:\Delta ABC::\Delta A'D'C':A'B'C'$. Now the bisector of a dihedral angle of a tetrahedron divides the opposite edge into segments proportional to the adjacent faces. Hence the plane $C'A'E$ is the bisector of dihedral angle $D'-C'A'-B'$.

The centroid of ΔABD and ΔBDC lies on $C'A'$, so the centroid, G, of the four faces of the tetrahedron $ABCD$ lies in the plane $C'A'E$. Similarly G lies on each of the dihedral angle bisectors of the tetrahedron $A'B'C'D'$, so G is the center of the inscribed sphere of this medial tetrahedron. That is, the center of mass of the surface of a tetrahedron is the center of its Spieker sphere.

Also solved by the *Proposer*.

*Johnson, *Modern Geometry*, p. 226, (1929).

No. 254. Proposed by *M. S. Robertson*, Rutgers University.

What is the value of

$$\operatorname*{Lim\ superior}_{n\to\infty} \operatorname*{Max}_{0\leqq\Theta\leqq\pi/3} \left|\frac{\cos n\Theta}{\cos\Theta}\right| ?$$

Solution by *Albert Farnell*, Louisiana State University.

The minimum absolute value of the denominator is $\frac{1}{2}$, and the maximum absolute value of the numerator is 1. For $\Theta=\pi/3$ and for every n a multiple of 3, these values are actually attained. Hence the desired limit superior is 2.

Solution by the *Proposer*.

The statement of the problem was supposed to include the restriction that n runs through those positive integers for which $n\not\equiv 0 \bmod 3$. In this case, let

$$f(\Theta)=\frac{\cos n\Theta}{\cos\Theta}, \quad \mu_n=\operatorname*{Max}_{0\leqq\Theta\leqq\pi/3}|f(\Theta)|.$$

For $n\not\equiv 0 \bmod 3$, $|f(\Theta)|=1$ at either end of the given interval. For $n>2$, there is a value of $\Theta>0$ for which $|f(\Theta)|=\mu_n>1$. This value Θ_n of Θ is a root of the equation $f'(\Theta)=0$ which is

$$(1) \qquad n\tan n\Theta=\tan\Theta.$$

We may then write

$$\mu_n^2=\left(\frac{\cos n\Theta_n}{\cos\Theta_n}\right)^2=\frac{\sec^2\Theta_n}{1+\tan^2 n\Theta_n}=\sec^2\Theta_n(1+n^{-2}\tan^2\Theta_n)^{-1}>1.$$

Thus $\operatorname*{lim\ superior}_{n\to\infty}\mu_n=\operatorname*{lim\ superior}_{n\to\infty}\sec\Theta_n$. Again, $f(\Theta)$ vanishes in the given interval for $\Theta=\pi(2k+1)/2n$ where k is any non-negative integer not greater than $[(2n-3)/6]$. Putting $k=[(2n-3)/6]=(2n-3)/6-\alpha$, $0\leqq\alpha<1$, we have $f(\Theta)=0$ for $\Theta=\pi/3-\pi\alpha/n$ and similarly, if $n\geqq 5$, $f(\Theta)=0$ for $\Theta=\pi/3-\pi(\alpha+1)/n$. By Rolle's theorem $f'(\Theta)$ vanishes for some value Θ_n lying between these two values of Θ at which $f(\Theta)=0$. Thus as n increases without limit equation (1) has a solution Θ_n arbitrarily near to $\pi/3$, whence

$$\operatorname*{lim\ superior}_{n\to\infty}\mu_n=\sec(\pi/3)=2.$$

Also solved by *Johannes Mahrenholz*.

No. 258. Proposed by *E. P. Starke,* Rutgers University.

When n is a positive integer, the coefficients in the binomial theorem may be divided into five groups as follows:

(1) ${}_nC_0, {}_nC_5, {}_nC_{10}, {}_nC_{15}, \cdots;$ (2) ${}_nC_1, {}_nC_6, {}_nC_{11}, {}_nC_{16}, \cdots;$

(3) ${}_nC_2, {}_nC_7, {}_nC_{12}, {}_nC_{17}, \cdots;$ (4) ${}_nC_3, {}_nC_8, {}_nC_{13}, {}_nC_{18}, \cdots;$

(5) ${}_nC_4, {}_nC_9, {}_nC_{14}, {}_nC_{19}, \cdots.$

Prove: if the terms in each set are added, the five sums will have just three distinct values; the three possible differences of distinct sums are three consecutive terms of the Fibonacci series,

$$1, 1, 2, 3, 5, 8, 13, \cdots, \text{ in which } a_{n+1} = a_n + a_{n-1}.$$

Solution by *Johannes Mahrenholz,* Gottbus, Germany.

Putting ${}_nS_j$, $j=0$, 1, 2, 3, 4, for the sum of terms of given set numbered $(j+1)$, we have at once

$$(1) \qquad {}_nS_j = {}^1/_5 \sum_{k=0}^{4} (2 \cos k\pi/5)^n \cos k(n-2j)\pi/5.$$

By using $\cos \pi/5 = (\sqrt{5}+1)/4 = -\cos 4\pi/5$ and $\cos 2\pi/5 = (\sqrt{5}-1)/4 = -\cos 3\pi/5$, we may reduce (1) to the form

$${}_nS_j' = \{2^n + 2^{1-n}[(\sqrt{5}+1)^n \cos(n-2j)\pi/5 + (\sqrt{5}-1)^n \cos 2(n-2j)\pi/5]\}/5.$$

Thus we have without difficulty

$${}_nS_j - {}_nS_j = 2^{2-n}\{(\sqrt{5}+1)^n[\sin(j'-j)\pi/5]\,[\sin(n-j-j')\pi/5]$$
$$+ (\sqrt{5}-1)^n[\sin 2(j'-j)\pi/5]\,[\sin 2(n-j-j')\pi/5]\}/5.$$

If $n \equiv j+j'$ mod 5, this difference is evidently zero. For other values of j and j', the difference reduces to a_{n-1}, a_n or a_{n+1}. For example if $n \equiv 0$ mod 5 and $j=1$, $j'=2$, we obtain

$$|{}_nS_1 - {}_nS_2| = \{(\sqrt{5}+1)^n - (1-\sqrt{5})^n\}/2^n\sqrt{5}$$

which is a known expression for a_n.

Editor's Note: Some of the numerical calculations indicated in the above solution may be avoided if we employ the method of mathematical induction. Thus, from the computed values of ${}_nS_j$ for the first several values of n, we observe

(A) for $n=2k$: ${}_nS_k$ is the largest sum; ${}_nS_j = {}_nS_k$ if and only if $j+k \equiv n$ mod 5; ${}_nS_k - 1 = {}_nS_{k+1} = {}_nS_k - a_{2k-1}$; ${}_nS_{k+2} = {}_nS_{k+3} = {}_nS_k - a_{2k+1}$.

(B) for $n=2k-1$: ${}_nS_{k-1}={}_nS_k$ is the largest sum; ${}_nS_j={}_nS_k$ if and only if $j+k\equiv n \bmod 5$; ${}_nS_{k+1}={}_nS_{k+3}={}_nS_{k-1}-a_{2k-1}$; ${}_nS_{k+2}={}_nS_{k-1}-a_{2k}$.

In this, it is understood that ${}_nS_k$ stands for ${}_nS_i$ where i is the least non-negative residue of k mod 5.

Since ${}_nC_0={}_{n+1}C_0$ and ${}_nC_r+{}_nC_{r+1}={}_{n+1}C_{r+1}$, we must have ${}_nS_r+{}_nS_{r+1}={}_{n+1}S_{r+1}$. Using also $a_{n+1}=a_n+a_{n-1}$, it is easy to show that if properties (A) and (B) hold for any value of n they hold also for $n+1$. These results evidently include the properties which were to be proved.

Analogous properties hold if the binomial coefficients are arranged in three sets: see problem E 300 in the American Mathematical Monthly for May, 1938. If they are arranged in four sets it is easy to show that two sums are always equal, and the other two are equal for odd n but unequal for even n; the differences between the sums are always powers of 2. The property of equality of certain sums holds for any number of sets, but the facts about the differences of sums are difficult to generalize beyond the case of five sums.

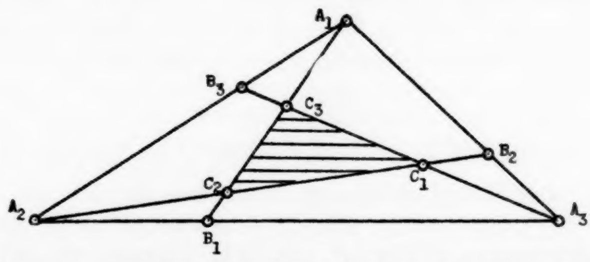
PROPOSALS

No. 281. Proposed by *Robert C. Yates*, University of Maryland.

Given a circle, C, together with its center, and a line, L, both in a plane. With the straightedge alone construct a perpendicular to a given point, P, of the line.

No. 282. Proposed by *M. Kac*, Lwów, Poland.*

Given the triangle A_1, A_2, A_3, with B_1, B_2, B_3 as points trisecting the sides.



Let the intersections of the lines A_iB_i be C_1, C_2, C_3. Show that

$$7s=S$$

where s is the area of $C_1C_2C_3$ and S is the area of $A_1A_2A_3$.

*This was proposed as a question in the entrance examination to the engineering school of St. Peterburg in 1912. See *H. Steinhaus*, Mathematical Snapshots, Stechert, (1938).

No. 283. Proposed by *H. T. R. Aude*, Colgate University.

Find a sequence of triads of integers such that when taken as sides of triangles these will approach an isosceles triangle with an angle of 120°.

No. 284. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Find an analogous formula for a tetrahedron that corresponds to the Law of Cosines for a triangle.

No. 285. Proposed by *Albert Farnell*, Louisiana State University.

Prove that the sum to n terms of the series whose nth term is $n\cos(2an+b)$ is equal to

$$\frac{n\sin(2an+a+b)}{2\sin a}-\frac{\sin an\sin(an+b)}{2\sin^2 a}.$$

No. 286. Proposed by *V. Thébault*, Le Mans, France.

Let M be an arbitrary point of the circle circumscribing the square $ABCD$. Let A_1, B_1, C_1, D_1 be the orthogonal projections of M upon AB, BC, CD, DA; and A_2, B_2, C_2, D_2 the projections of M upon $A_1B_1, B_1C_1, C_1D_1, D_1A_1$. Show that A_2, B_2, C_2, D_2 are collinear.

No. 287. Proposed by *Yudell Luke*, University of Illinois.

Let the digits of a 3-digit number N be reversed to form N'. Put D for the difference $|N-N'|$, and let the digits of D be reversed to form D'. (D is considered to have three digits: thus if $D=099$, $D'=990$.) Then, if $D\neq 0$, $D+D'$ has the fixed value 1089. This well known fact is easily proved. What values may $D+D'$ take corresponding to N a 4-digit number?

No. 288. Proposed by *L. E. Bush*, College of St. Thomas, Minnesota.

Part I. Let D and A' be any two points on side AC of triangle ABC and E any point on BD. Let CE intersect AB at M, AE intersect BC at N, DN intersect $A'B$ at M', and MM' intersect AC at D'. Prove that D and D' separate A and A' harmonically.

Part II. Use the above theorem to prove that if in the triangle ABC, D is the foot of the perpendicular from B to AC, and if M and N are points on AB and CB respectively, such that AN and CM intersect on BD, then BD bisects angle MDN.*

*Altshiller-Court, *College Geometry*, p. 92.

No. 289. Proposed by *V. Thébault*, Le Mans, France.

In the equations

$$X^2=Y^2+Z^2 \quad \text{and} \quad x^2=y^2+z^2,$$

the letters represent positive integers, and Y and Z are relatively prime, as also y and z. Show that the numbers

$$Xx+Yy+Zz \quad \text{and} \quad Xx+Yz+Zy$$

are, the one a square, the other twice a square.

No. 290. Proposed by *J. W. Peters*, University of Illinois.

Given three mutually orthogonal circles C_1, C_2, C_3 in a plane, with centers c_1, c_2, c_3. The point n_3 is the inverse of c_1 in C_2 and the inverse of c_2 in C_1; n_2 is the inverse of c_1 in C_3 and the inverse of c_3 in C_1; n_1 the inverse of c_2 in C_3 and c_3 in C_2. If P is the radical center of C_1, C_2, C_3, prove that P is the inverse of n_1 in C_1, the inverse of n_2 in C_2, and the inverse of n_3 in C_3.

"The reason why geometry is not so difficult as algebra, is to be found in the less general nature of the symbols employed. In algebra a general proposition respecting numbers is to be proved. Letters are taken which may represent any of the numbers in question, and the course of the demonstration, far from making any use of a particular case, does not even allow that any reasoning, however general in its nature, is conclusive unless the symbols are as general as the arguments."

From De Morgan's "The Study of Mathematics." The Open Court Publishing Company.

Bibliography and Reviews

Edited by
P. K. SMITH and H. A. SIMMONS

A Short Course in Trigonometry. Revised Edition. By James G. Hardy, The Macmillan Company, New York, 1938. lx+152. With the Macmillan Tables, reset, xvi+143. $2.25.

This book is a revision of the first edition of the same title published in 1932. The author says of the revision: "The order of topics in Chapter VI (Variation of the Trigonometric Functions) has been changed and the sections on Inverse Functions (in Chapter IX) rewritten. Instead of reserving the topic of Trigonometric Equations until late in the course, simple equations are introduced early and they are used freely throughout".

The general plan of organization is seen from the chapter titles and pagination: I, *The Trigonometric Functions,* 1-22; II, *General Definitions,* 23-30; III, *Relations Between the Trigonometric Functions,* 31-39; IV, *Reduction to Acute Angles,* 40-46; V, *Solution of Oblique Triangles,* 47-60; VI, *Variation of the Trigonometric Functions,* 61-73; VII, *Radian Measure,* 74-85; VIII, *Functions of Several Angles,* 86-99; IX, *The Solution of Trigonometric Equations and Inverse Functions,* 100-109; X, *Logarithmic Solution of Triangles,* 110-129; XI, *Logarithms,* 130-141.

With a few exceptions, all of the customary material of plane trigonometry is to be found in this book, either in exposition or problems. There is inadequate discussion of approximate computation and significant figures; the graphs of the inverse trigonometric functions are not mentioned, and there are few identities in the familiar equation form. The following topics are included in exercises but not in exposition: *functions of negative angles* (used in the exposition on p. 90), *reduction formulas based on 90° and 270°, graph and period of the secant and cosecant,* and *area of sector and segment.* Some attention is given to parametric equations in which the parameter is an angle. It will be seen that the difference in content between this "short" course and the usual one is not great.

The exposition is generally lucid without being verbose. The book is addressed to the student rather than the instructor, and students frequently are given specific suggestions or warnings on points of special

difficulty. The exercises, "of which there are more than fourteen hundred, are new with the exception of a very few which occur in all trigonometries and which have become classic." Answers are given for the odd-numbered problems; in the case of problems on the solution of triangles, answers based on both four and five place tables are given. The reviewer has made no check on the accuracy of the answers.

The format and printing are up to the high standard maintained by the publisher. Figures are plentiful and are well-drawn. The few typographical errors are not of a nature to cause difficulty. The tables are the complete Macmillan Tables, recently reset in a new type face, and are more than ample for a trigonometry course.

The reviewer noted two errors other than typographical. On p. 11, Exercise IV, a positive angle is specified and the notion of directed angle has not yet been introduced. On p. 69 the implication of the statement concerning the variation of the tangent is that it has a period of 360°.

Some teachers will dislike the near absence of trigonometric identities in equation form. The author chooses to list one member of the identity and to require its transformation to a second given form. The ambiguous case in oblique triangles will arise naturally in §27 and in §63, but it is not mentioned or treated until §68.

Unfortunately, reading a text is not an adequate criterion of its teachability. This appears to be a teachable text, suitable for use in the usual course in plane trigonometry provided careful attention is given to the problems in certain places.

University of Illinois H. W. BAILEY.

College Algebra. Second Revised Edition. By Louis J. Rouse. John Wiley & Sons, Inc., New York, 1939. xiii+462 pages.

According to the preface, "In this Edition a new chapter on compound interest and annuities has been added. . . . The chapter on probability is new. The chapter on determinants has been completely re-written. In Chapter XII a section on harmonic progressions . . . has been added. The chapter on inequalities has been materially strengthened. In the chapter on the theory of equations the section devoted to the treatment of zero and infinite roots is new. And finally a chapter on partial fractions. . . has been added. The various chapters are quite independent of each other. Special attention has been given to the needs of technical students and others who intend to continue their study of mathematics at least through elementary calculus".

The writer agrees with all of the statements just made and notes the following good features of the book: It is a rather complete text-book of college algebra; it has an outstanding problem list, including far more word-problems, sometimes called *story problems,* than one finds in most texts on college algebra; the arrangement of the material is good; and the format is splendid.

In some places, the text needs revision. Examples of such places are as follows: on pages 62 and 122, linear and quadratic equations, respectively, are incorrectly defined; on page 94, "the cube root" should be replaced by "a cube root" and similar mistakes are frequent in the text; on page 103, the author writes $\sqrt{ab}=\sqrt{a}\,\sqrt{b}$ without excluding the case in which n is even and both a and b are negative; and on page 336 an essential induction is omitted.

Nevertheless, considered as a whole, the book is very teachable and it should afford many college teachers a pleasant change.

Northwestern University H. A. SIMMONS.

The Nature of Proof. By Harold P. Fawcett. Thirteenth Yearbook of the National Council of Teachers of Mathematics. Bureau of Publications, Teachers College, Columbia University, 1938.

During the last thirteen years, the National Council of Teachers of Mathematics has published a number of yearbooks which should be a part of the private library of every high school teacher of mathematics. The most recent volume of this series is related to the problem of improving the teaching of high school geometry. It will be welcomed by the teachers of that subject.

One of the major values claimed for geometry is the training it offers in correct and efficient methods of thinking. Dr. Fawcett presents in detail the procedure that he has followed in aiding pupils to attain this value. The book is divided into six chapters, but the part that will be most appreciated by the teachers is Chapter III. He illustrates and describes the steps in training pupils to think clearly, not only in geometric but also in non-mathematical life situations.

The majority of the pupils in the group receiving this training belonged to the eleventh grade. Some tenth grade pupils and a few ninth grade pupils completed the group.

The first step in the procedure was to lead the pupils by means of non-mathematical life situations to the recognition that whenever careful thinking is required there should be no vagueness or ambiguity as to the meaning of the words, concepts, and assumptions that are

involved. In other words, such terms and concepts must be defined. This was shown to be important because any conclusion a pupil may reach will depend on these definitions and assumptions. Often it was found that terms could not be defined and would have to remain undefined.

The second step was to establish the fact that the same caution as to clearness of defined and undefined terms applies also to discussions relating to space, both two-dimensional and three-dimensional.

Sometimes the order of the two foregoing steps was reversed. That is, geometry was used to introduce the pupil to methods of thinking, and the methods were then applied to non-mathematical situations. This phase of the work is probably the best contribution of Dr. Fawcett's study. Numerous excellent examples are given to show how geometry may be taught so as to train pupils to think clearly in life situations, which is generally accepted as a major objective of the teaching of mathematics.

Another interesting phase of Chapter III is the description of experimental methods of establishing geometric facts. Definitions are made to grow out of experiences. They are first formulated by the pupil in his own way. Later the statements are refined by class and teacher. Often the pupil selects from the facts that have been discovered those which he is going to prove.

Chapter I acquaints the reader with the purpose of the study and with the problem it aims to solve. It is really a defense of one of the major objectives of geometry whose value has never been questioned. The author seems to magnify the importance of one objective by lessening the importance of other generally accepted objectives. His interpretations of certain committee reports and comments of other writers lead him to the conclusion that "demonstrative geometry is no longer justified on the ground that it is necessary for the purpose of giving students control of useful geometric knowledge." The National Committee correctly names this as one of the important objectives of geometry. We live in a world of geometry and pupils should become efficient in applying geometric knowledge to life situations just as they should be able to use correct reasoning processes in life situations. Moreover, in his future mathematics the pupil will have to know the facts taught in demonstrative geometry but he will probably never be called upon to prove them.

Chapters II and V describe the author's method of research and the evaluation of achievement. Twenty-five pupils took part in the experiment. The average I. Q. was 115 with a range of 91-133. Twenty-two of the group had a distinctly negative attitude toward studying any more mathematics.

The Ohio *Every-Pupil-Test** was administered after the year's work. This test measures largely the acquisition of geometric content acquired. The median score of the group was 52.0 which compares very well with the norm of 36.5. As measured by this test the class gave a good account of itself as to knowledge of geometric subject matter.

A *Nature-of-Proof* test designed by the author was given before and after the year's work. The median score was raised from 16.7 to 24.2. This was a very good showing when compared with 3 classes in demonstrative geometry which also took the test. The changes of these classes were respectively, from 13.0 to 14.0, 14.3 to 14.1, and 13.8 to 12.8.

It is to be regretted that the pupils' ability to prove theorems or exercises was not evaluated.

Reactions of pupils, parents, and student observers of classes were reported. Such reactions although not regarded as valid measures of achievement are interesting. Apparently the author succeeded in changing the original negative attitude of the pupils to one of interest and enthusiasm.

University of Chicago E. R. BRESLICH.

Trigonometry. By Howard K. Hughes and Glen T. Miller. John Wiley & Sons, Inc., New York, 1939. Text, 189 pages: Tables, 79 pages. Published with or without tables. Contains index and answers to odd-numbered exercises.

According to the authors, this book is intended for use in a first course in trigonometry for advanced high school or first-year college students. Since they believe that numerical trigonometry is more easily grasped by the average beginner than is analytical trigonometry, they have placed the solution of triangles early in the text and have postponed the analytical trigonometry until the later chapters. Chapters I-VII carry the student through the solution of oblique triangles without the use of logarithms. Chapters IX and X cover logarithms and the logarithmic solution of triangles. The remaining chapters treat radian measure, fundamental identities, equations, multiple angles, variation, line values, graphs, inverse functions; and a final chapter is devoted to an introduction to spherical trigonometry.

Although the authors have placed the formal treatment of analytical trigonometry in the latter part of the text, they have anticipated

*Every-Pupil-Test, State Department of Education, Columbus, Ohio, 1936.

many of the formulas by including exercises which require the verification of the formulas for particular angles. On the whole, the exercises seem well-graded and adequate in number. A pleasing feature is the inclusion of quite a number of exercises which are literal in character and require the student to focus his attention on the derivation of functional relationships. On the other hand, in a list of more than a hundred exercises in the chapter on fundamental identities there is only one exercise which involves any multiple angle, such as $3B$ or $A/2$.

The explanations are clear and the general treatment is satisfactory. The reviewer approves the arrangement of material because he has just finished a course where this general order of top cs was used with considerable satisfaction. Perhaps the chapter on logarithms is too brief; and this criticism might be extended to the work on graphs. However, this is a matter of individual preference and habit. The same is true of the reviewer's reaction toward the manner of derivation of certain of the formulas.

The typography is pleasing and remarkably free from error. Figures are well arranged and carefully labeled.

In the opinion of the reviewer, the text is well adapted to the purpose for which it has been written and should prove very teachable.

Carnegie Institute of Technology EDWIN G. OLDS.

Announcement!

Beginning with Volume XIV, October 1, 1939, the subscription price of NATIONAL MATHEMATICS MAGAZINE will be increased to $2.00 per volume, or per year.

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Editor and Manager
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